Q1

$$\text{area}_w = \text{area}_u + \text{area}_v$$

$$\frac{1}{2}xy\sin(A+B) = \frac{1}{2}(1)(x)\sin A + \frac{1}{2}(1)y\sin B$$

$$xy\sin(A+B) = x\sin A + y\sin B$$

$$\sin(A+B) = \frac{1}{y}\sin A + \frac{1}{x}\sin B$$

But by SOHCAHTOA, $\cos A = \frac{1}{x}$ and $\cos B = \frac{1}{y}$

Therefore

$$\sin(A+B) \equiv \sin A\cos B + \sin B\cos A$$

SOHCAHTOA is only valid in right-angled triangles, and this proof uses the SOHCAHTOA results $\cos A = 1/x$ and $\cos B = 1/y$. In a right-angled triangle all the non-right angles are acute, therefore A and B are acute. It follows that the proof given here is only valid for acute A and B.

Q2

Prove the identity

$$\tan 4\theta \equiv \frac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$$

[4]

$A = 2\theta$

$A = \theta$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A} \quad \left[\text{Double angle formula}\right]$$

$$\tan 4\theta \equiv \frac{2\tan 2\theta}{1-\tan^2 2\theta}$$

$$\equiv \frac{2\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}{1-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)^2}$$

$$= \frac{\frac{4\tan\theta}{1-\tan^2\theta}}{\frac{(1-\tan^2\theta)^2-(2\tan\theta)^2}{(1-\tan^2\theta)^2}}$$

$$= \frac{4\tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta)^2-(2\tan\theta)^2}$$

$$= \frac{4\tan\theta(1-\tan^2\theta)}{1-2\tan^2\theta+\tan^4\theta-4\tan^2\theta}$$

$$= \frac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$$

Therefore

$$\tan 4\theta \equiv \frac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$$

Q3

Prove the identity

$$-16\cot 2\theta\,\mathrm{cosec}^3 2\theta \equiv \sec^4\theta - \mathrm{cosec}^4\theta$$

[5]

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$$-16\cot 2\theta\,\mathrm{cosec}^3 2\theta \equiv -16\left(\frac{\cos 2\theta}{\sin 2\theta}\right)\left(\frac{1}{\sin 2\theta}\right)^3$$

$$= \frac{-16\cos 2\theta}{\sin^4 2\theta}$$

$$\equiv \frac{-16(\cos^2\theta - \sin^2\theta)}{(2\sin\theta\cos\theta)^4}$$

$$= \frac{-\cancel{16}(\cos^2\theta - \sin^2\theta)}{\cancel{16}\sin^4\theta\cos^4\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\sin^4\theta\cos^4\theta}$$

$\sin^2\theta + \cos^2\theta \equiv 1$

$$\equiv \frac{(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)}{\sin^4\theta\cos^4\theta}$$

$$= \frac{\sin^4\theta - \cos^4\theta}{\sin^4\theta\cos^4\theta}$$

$$= \frac{1}{\cos^4\theta} - \frac{1}{\sin^4\theta}$$

$$\equiv \sec^4\theta - \mathrm{cosec}^4\theta$$

Therefore

$$-16\cot 2\theta\,\mathrm{cosec}^3 2\theta \equiv \sec^4\theta - \mathrm{cosec}^4\theta$$

Q4

Show that

$$\frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin\left(\theta - \frac{\pi}{2}\right)} \equiv \tan\theta - 1$$

[4]

$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ [Compound angle formula]

$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$ [Compound angle formula]

$$\frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin\left(\theta - \frac{\pi}{2}\right)} = \frac{\sqrt{2}\left(\cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4}\right)}{\sin\theta\cos\frac{\pi}{2} - \cos\theta\sin\frac{\pi}{2}}$$

$$= \frac{\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta\right)}{(0)\sin\theta - (1)\cos\theta}$$

$$= \frac{\cos\theta - \sin\theta}{-\cos\theta}$$

$$= \frac{\sin\theta - \cos\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} - 1$$

$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

$$\equiv \tan\theta - 1$$

Therefore

$$\frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin\left(\theta - \frac{\pi}{2}\right)} \equiv \tan\theta - 1$$

Q5a

(a) Show that

$$\sin 3\theta \equiv 3\sin\theta\cos^2\theta - \sin^3\theta$$

[4]

(b) Hence, or otherwise, show that

$$\frac{\cos 3\theta - \cos\theta}{\sin 3\theta \sin\theta} \equiv \frac{4\cos\theta}{1 - 4\cos^2\theta} \qquad \theta \neq k\pi$$

[5]

$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ [Compound angle formula]

a) $\sin 3\theta = \sin(2\theta + \theta)$, so

$$\begin{aligned}
\sin 3\theta &\equiv \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\
&\equiv (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta \\
&= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta \\
&= 3\sin\theta\cos^2\theta - \sin^3\theta
\end{aligned}$$

Therefore

$$\sin 3\theta \equiv 3\sin\theta\cos^2\theta - \sin^3\theta$$

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

Q5b

(a) Show that

$$\sin 3\theta \equiv 3\sin\theta\cos^2\theta - \sin^3\theta$$

[4]

(b) Hence, or otherwise, show that

$$\frac{\cos 3\theta - \cos\theta}{\sin 3\theta \sin\theta} \equiv \frac{4\cos\theta}{1 - 4\cos^2\theta} \qquad \theta \neq k\pi$$

[5]

$\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ This identity may be derived from the compound angle formulae

b) $\cos 3\theta - \cos\theta \equiv -2\sin\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)$

$= -2\sin 2\theta\sin\theta$

So

$$\begin{aligned}
\frac{\cos 3\theta - \cos\theta}{\sin 3\theta\sin\theta} &\equiv \frac{-2\sin 2\theta\,\cancel{\sin\theta}}{(3\sin\theta\cos^2\theta - \sin^3\theta)\cancel{\sin\theta}} \\
&\equiv \frac{-2(2\sin\theta\cos\theta)}{3\sin\theta\cos^2\theta - \sin^3\theta} \\
&= \frac{-4\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}(3\cos^2\theta - \sin^2\theta)} \\
&= \frac{-4\cos\theta}{3\cos^2\theta - \sin^2\theta} \\
&= \frac{4\cos\theta}{\sin^2\theta - 3\cos^2\theta} \\
&\equiv \frac{4\cos\theta}{(1-\cos^2\theta) - 3\cos^2\theta} \\
&= \frac{4\cos\theta}{1 - 4\cos^2\theta}
\end{aligned}$$

Therefore

$$\frac{\cos 3\theta - \cos\theta}{\sin 3\theta\sin\theta} \equiv \frac{4\cos\theta}{1 - 4\cos^2\theta}$$

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\sin^2\theta \equiv 1 - \cos^2\theta$

Alternative method

$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ [Compound angle formula]

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

$\cos 3\theta \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (1-2\sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$= \cos\theta - 4\sin^2\theta\cos\theta$

So $\dfrac{\cos 3\theta - \cos\theta}{\sin 3\theta\sin\theta} \equiv \dfrac{(\cos\theta - 4\sin^2\theta\cos\theta) - \cos\theta}{(3\sin\theta\cos^2\theta - \sin^3\theta)\sin\theta}$

$\equiv \dfrac{-4\cancel{\sin^2\theta}\cos\theta}{\cancel{\sin^2\theta}(3\cos^2\theta - \sin^2\theta)}$

$= \dfrac{-4\cos\theta}{3\cos^2\theta - \sin^2\theta}$

$\equiv \dfrac{-4\cos\theta}{3\cos^2\theta - (1-\cos^2\theta)}$ $\quad \sin^2\theta \equiv 1 - \cos^2\theta$

$= \dfrac{-4\cos\theta}{4\cos^2\theta - 1}$

$= \dfrac{4\cos\theta}{1 - 4\cos^2\theta}$

Therefore

$$\frac{\cos 3\theta - \cos\theta}{\sin 3\theta\sin\theta} \equiv \frac{4\cos\theta}{1-4\cos^2\theta}$$

Q6

Show that

$$4\cos^2\left(x - \frac{\pi}{6}\right) \equiv 3 - 2\sin^2 x + \sqrt{3}\sin 2x$$

[5]

$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$ [Compound angle formula]

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\sin^2 A + \cos^2 A \equiv 1$

$4\cos^2\left(x - \frac{\pi}{6}\right) \equiv 4\left(\cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6}\right)^2$

$= 4\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right)^2$

$= 4\left(\frac{3}{4}\cos^2 x + \frac{\sqrt{3}}{2}\sin x\cos x + \frac{1}{4}\sin^2 x\right)$

$= 3\cos^2 x + \sin^2 x + \sqrt{3}(2\sin x\cos x)$

$\equiv 3(1-\sin^2 x) + \sin^2 x + \sqrt{3}\sin 2x$

$= 3 - 3\sin^2 x + \sin^2 x + \sqrt{3}\sin 2x$

$= 3 - 2\sin^2 x + \sqrt{3}\sin 2x$

Therefore

$$4\cos^2\left(x - \frac{\pi}{6}\right) \equiv 3 - 2\sin^2 x + \sqrt{3}\sin 2x$$

Q7

Show that

$$\tan\left(\frac{2x+\pi}{4}\right) \equiv \sec x + \tan x$$

[6]

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1-\tan A \tan B}$$ [Compound angle formula]

$\tan\left(\frac{2x+\pi}{4}\right) = \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$, so

$\tan\frac{\pi}{4} = 1$

$$\tan\left(\frac{2x+\pi}{4}\right) = \frac{\tan\frac{x}{2}+\tan\frac{\pi}{4}}{1-\tan\frac{x}{2}\tan\frac{\pi}{4}} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

$$\equiv \frac{1+\frac{\sin(x/2)}{\cos(x/2)}}{1-\frac{\sin(x/2)}{\cos(x/2)}} = \frac{\frac{\cos(x/2)+\sin(x/2)}{\cancel{\cos(x/2)}}}{\frac{\cos(x/2)-\sin(x/2)}{\cancel{\cos(x/2)}}} = \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}$$

$$= \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} \times \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$$

$$= \frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$$

$A = \frac{x}{2}$

$$\equiv \frac{1+\sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$

$$\equiv \sec x + \tan x$$

Therefore

$$\tan\left(\frac{2x+\pi}{4}\right) \equiv \sec x + \tan x$$

$$\sin^2 A + \cos^2 A \equiv 1$$

$$\sin 2A \equiv 2\sin A\cos A$$ [Double angle formula]

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$ [Double angle formula]

Q8

Show that

$$\frac{1}{\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)^2} + \frac{1}{\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)^2} \equiv 4\,\text{cosec}^2\left(2\theta + \frac{\pi}{3}\right)$$

[9]

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$ [Compound angle formula]

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$ [Compound angle formula]

$$\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \cos\frac{\pi}{6}\cos\theta - \sin\frac{\pi}{6}\sin\theta \equiv \cos\left(\theta+\frac{\pi}{6}\right)$$

$$\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = \cos\frac{\pi}{6}\sin\theta + \sin\frac{\pi}{6}\cos\theta \equiv \sin\left(\theta+\frac{\pi}{6}\right)$$

$$\frac{1}{\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)^2} + \frac{1}{\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)^2} \equiv \frac{1}{\cos^2\left(\theta+\frac{\pi}{6}\right)} + \frac{1}{\sin^2\left(\theta+\frac{\pi}{6}\right)}$$

$$= \frac{\sin^2\left(\theta+\frac{\pi}{6}\right)+\cos^2\left(\theta+\frac{\pi}{6}\right)}{\cos^2\left(\theta+\frac{\pi}{6}\right)\sin^2\left(\theta+\frac{\pi}{6}\right)}$$

$\sin^2 A + \cos^2 A \equiv 1$

$$\equiv \frac{1}{\left(\cos\left(\theta+\frac{\pi}{6}\right)\sin\left(\theta+\frac{\pi}{6}\right)\right)^2}$$

$$\equiv \frac{1}{\left(\frac{1}{2}\sin\left(2\theta+\frac{\pi}{3}\right)\right)^2} = \frac{1}{\frac{1}{4}\sin^2\left(2\theta+\frac{\pi}{3}\right)}$$

$$= \frac{4}{\sin^2\left(2\theta+\frac{\pi}{3}\right)} \equiv 4\,\text{cosec}^2\left(2\theta+\frac{\pi}{3}\right)$$

Therefore

$$\frac{1}{\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)^2} + \frac{1}{\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)^2} \equiv 4\,\text{cosec}^2\left(2\theta+\frac{\pi}{3}\right)$$

$$\sin 2A \equiv 2\sin A\cos A$$ [Double angle formula]

$A = \theta + \frac{\pi}{6}$